Towards Closing the Trust Gap in the SAT Certification of Ramsey Numbers R(3,8) and R(3,9)

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Chapter 1

Introduction

This project aims to formalize some results about Ramsey theory by Graver and Yackle, which were used in the SAT certification for Ramsey numbers $R(3, 8)$ and $R(3, 9)$ by Zhengyu Li.

Chapter 2

Sym2

Definition 1 (decBall)**.**

Definition 2 (decBex)**.**

Definition 3 (toOrderedPair)**.**

Lemma 4 (exists_mem_pair)**.**

Proof. □

Lemma 5 (toOrderedPair_repr)**.**

Proof. □

Lemma 6 (toOrderedPair_inj)**.**

Proof. □

Lemma 7 (toOrderedPair_IsDiag_of_eq)**.**

Proof. □

Lemma 8 (isDiag_of_subsingleton)**.**

Proof. □

Chapter 3

Bron–Kerbosch Algorithm

Lemma 9 (length_wf)**.**

Proof. □

Definition 10 (Bron_Kerbosch_full)**.**

Lemma 11 (Bron_Kerbosch_maximal_clique)**.**

Proof. □

Definition 12 (Bron_Kerbosch)**.**

Lemma 13 (Bron_Kerbosch_correct)**.**

Proof. □

Lemma 14 (Bron_Kerbosch_maximum_clique)**.**

Proof. □

Lemma 15 (Bron_Kerbosch_cliqueNum)**.**

Proof. □

Chapter 4

SimpleGraph

Lemma 16 (induce_compl)**.**

Proof. □

Theorem 17 (IsIndepSet.subset)**.**

Proof. □

Theorem 18 (Iso.isNIndepSet_one)**.**

Proof. □

Lemma 19 (Iso.IsClique)**.**

Proof. □

Lemma 20 (Iso.IsIndepSet)**.**

Proof. □

Lemma 21 (Iso.IsNClique)**.**

Proof. □

Lemma 22 (Iso.IsNIndepSet)**.**

Proof. □

Lemma 23 (Iso.cliqueNum)**.**

Proof. □

Lemma 24 (Iso.indepNum)**.**

Proof. □

Definition 25 (Iso.compl)**.**

Lemma 26 (fintype_cliqueNum_bddAbove)**.**

Proof. □

Lemma 27 (fintype_indepNum_bddAbove)**.**

Proof. □

Lemma 28 (cliqueNum_pos)**.**

Proof. □

Lemma 29 (indepNum_pos)**.**

Proof. □

Lemma 30 (cliqueNum_top)**.**

Proof. □

Lemma 31 (cliqueNum_bot)**.**

Proof. □

Lemma 32 (indepNum_bot)**.**

Proof. □

Lemma 33 (indepNum_top)**.**

Proof. □

Lemma 34 (exists_isNClique_of_le_cliqueNum)**.**

Proof. □

Lemma 35 (exists_isNIndset_of_le_indepNum)**.**

Proof. □

Lemma 36 (cliqueNum_lt_iff_cliqueFree)**.**

Proof. □

Lemma 37 (Embedding.indepNum_mono)**.**

Proof. □

Lemma 38 (Embedding.cliqueNum_mono)**.**

Proof. □

Chapter 5

Inequalities

Lemma 39 (Fin.val_three)**.**

Proof. □

Lemma 40 (σ_helper)**.**

Proof. □

Lemma 41 (s_i_vanish)**.**

Proof. □

Theorem 42 (R36ub)**.**

Theorem 43 (R36)**.**

Proof. □

Theorem 44 (R37ub)**.**

Theorem 45 (R37)**.**

Proof. □

Theorem 46 ($e3611 \geq 11$)**.**

Theorem 47 ($e3612 \geq 15$)**.**

Theorem 48 ($e3613 \geq 20$)**.**

Theorem 49 ($e3614 \geq 15$)**.**

Theorem 50 ($e3615 \geq 32$)**.**

Theorem 51 ($e3616 \geq 50$)**.**

Lemma 52 (R38lb)**.**

Proof. □

Lemma 53 ($e3718 \geq 36$)**.**

Proof. ☐

Theorem 54 ($e3719 \geq 44$)**.**

Lemma 55 ($e3720 \geq 50$)**.**

Proof. ☐

Lemma 56 ($e3721 \geq 59$)**.**

Proof. ☐

Lemma 57 ($e3721 \geq 59$)**.**

Proof. ☐

Theorem 58 (Graver's Ineq 9: $e3826 \geq 80$)**.**

Proof. ☐

Theorem 59 (Graver's Ineq 9: $e3826 \geq 80$)**.**

Proof. ☐

Theorem 60 (Graver's Ineq 10: $e3827 \geq 88$)**.**

Proof. ☐

Theorem 61 (Graver's Ineq 11: $e3828 \geq 99$)**.**

Proof. ☐

Lemma 62 (Gregular_of_isXY)**.**

Proof. ☐

Theorem 63 (R39Ineq)**.**

Proof. ☐

Theorem 64 (R39_36_graph_IsRegular8)**.**

Proof. ☐

Theorem 65 (R39_36_graph_has_R38_27)**.**

Proof. ☐

Chapter 6

Ramsey Theory

Definition 66 ((x, y)-graphs)**.** G is an (x, y)-graph if $x > C(G)$ and $y > I(G)$.

Definition 67 (Ramsey Number (Graver's version))**.**

$$R(x, y) = \max\{n \in \mathbb{N} \mid \exists G,\, G \text{ is an}(x, y)\text{-graph } \wedge\ |V(G)| = n\}$$

Theorem 68 (Graver's Lemma 1)**.** *G is an (x, y)-graph if and only if its complement G^c is a (y, x)-graph.*

Proof. □

Theorem 69 (noXYGraphIffRamseyGraphProp)**.**

Proof. □

Lemma 70 (isXYGraph_bddAbove)**.**

Proof. □

Lemma 71 (cardLERamseyOld)**.**

Proof. □

Lemma 72 (R3y_neighbor_Ind)**.**

Proof. □

Theorem 73 (GraphRamsey2RamseyOld)**.**

Proof. □

Theorem 74 (RamseyOld_2)**.**

Proof. □

Definition 75 (v_i)**.**

Definition 76 (s_i)**.**

Definition 77 (t_i)**.**

Definition 78 ($\sigma(G)$)**.**

Lemma 79 (sum_degree_eq_sum_over_degrees)**.**

Proof. ☐

Lemma 80 (num_of_vertices_eq_sum_over_degrees)**.**

Proof. ☐

Definition 81 (H_1)**.**

Definition 82 (H_2)**.**

Definition 83 (e_1)**.**

Definition 84 (e_2)**.**

Definition 85 ($H_1 H_2$ isomorphic)**.**

Lemma 86 (H_1_eq_bot_of_3y)**.**

Proof. ☐

Lemma 87 (H_1_cliqueNum_lt)**.**

Proof. ☐

Theorem 88 (Graver Lemma 2)**.** *For any (x, y, n)-graph G, every vertex v in G satisfies*

$$(n-1) - R(x, y-1) \leq deg(v) \leq R(x-1, y)$$

.

Proof. ☐

Lemma 89 (G_degreeCount_eq)**.**

Proof. ☐

Lemma 90 (G_vertCount_eq)**.**

Proof. ☐

Lemma 91 (H_1_degreeCount_eq)**.**

Proof. ☐

Lemma 92 (H_1_vertCount_eq)**.**

Proof. ☐

Theorem 93 (Graver's Prop 1)**.**

Proof. ☐

Lemma 94 (v_i_deg_swap)**.**

Proof. ☐

Theorem 95 (Graver's Prop 2)**.**

Proof. □

Definition 96 (e)**.**

Lemma 97 (exy0)**.**

Proof. □

Lemma 98 (e_2_ge_e)**.**

Proof. □

Theorem 99 (Graver's Prop 4)**.**

Proof. □